Some preliminaries:

<u>Def</u> Let $\mathcal{D}$ be an ∞-cat. $\mathcal{D}$ is <u>differentiable</u> if

a) $\mathcal{D}$ has finite limits

b) $\mathcal{D}$ has sequential colimits

c) the functor $\varinjlim : \mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{D}) \longrightarrow \mathcal{D}$

is left-exact. Read: seq. colimits commute w/ finite limits.

<u>Examples</u>: • Any ∞-topos (so spaces)

• Any stable ∞-cat (so spectra)

<u>Def</u> An ∞-cat will be called <u>good</u> if it has finite colimits & a terminal object.

<u>Thm</u> If $\mathcal{C}$ is good & $\mathcal{D}$ is diff'l, then the inclusion functor

$$\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

of n-excisive functors admits a left-exact left-adjoint P_n.

<u>Chain Rule</u>: If $G : \mathcal{D} \longrightarrow \mathcal{D}'$ is a functor between diff'l ∞-cats & G preserves finite limits & seq colims, then $\forall F : \mathcal{C} \longrightarrow \mathcal{D}$

$$P_n(G \circ F) \simeq G \circ P_n(F).$$

<squeeze in other chain rule>

go straight to multi.

<u>Def</u> A functor $F : \mathcal{C} \hookrightarrow \mathcal{D}$ is <u>n-reduced</u> if $P_{n-1}(F)$ is a final object of $\mathrm{Exc}^{n-1}(\mathcal{C}, \mathcal{D})$.

$F' : \mathcal{C}' \to \mathcal{C}^{\text{good}}$, F preserves pushouts + terminal objects $\Rightarrow P_n(F \circ F') \simeq P_n F \circ F'$

F is <u>n-homogeneous</u> if it is n-excisive & n-reduced.

Denote the corresponding ∞-cat by $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$

closed under sequential colimits.

<u>Notice</u>: $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ also contains the terminal functor.

$\Rightarrow$ if $\mathcal{D}$ is pointed, so is $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

<u>Prop</u> (Cor 7.1.2.8)

Let $\mathcal{C}$ be good & $\mathcal{D}$ pointed, and $n \geq 1$. Then $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is stable.

<u>Rmk</u> Let $Sp(\mathcal{D}) = \mathrm{Exc}_*(\mathcal{S}_*^{fin}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{D})$ be the stabilization of $\mathcal{D}$. $\forall\ K \in \mathcal{S}_*^{fin}$, we have

$$\begin{aligned} ev_K : Sp(\mathcal{D}) &\longrightarrow \mathcal{D} \\ X &\longmapsto X(K) \end{aligned}.$$

<u>Note</u> $Sp(\mathcal{D})$ is closed under finite limits and colimits in $\mathrm{Fun}(\mathcal{S}_*^{fin}, \mathcal{D})$

$\Rightarrow$ ev_K preserves finite limits & seq colims

$\forall\ F : \mathcal{C} \longrightarrow Sp(\mathcal{D})$,

$P_n(e_K \circ F) \simeq e_K \circ P_n(F)$

$\Rightarrow$ F is n-excisive $\Leftrightarrow$ each $e_K \circ F$ is

" " n-reduced $\Leftrightarrow$ " "

stability!

$\Rightarrow \mathrm{Homog}^n(\mathcal{C}, Sp(\mathcal{D})) \simeq Sp(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$

<u>Multivariable Calculus</u>

<u>Def</u> Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be ∞-cats w/ pushouts & let $\mathcal{D}$ have finite limits. Suppose $\vec{n} = (n_1, \ldots, n_m)$ is a seq. of non-negative integers. A functor $F : \prod_j \mathcal{C}_j \longrightarrow \mathcal{D}$ is <u>$\vec{n}$-excisive</u> if

- $\forall\ 1 \leq i \leq m$ & $\{X_j \in \mathcal{C}_j\}_{j \neq i}$, the functor

$$\mathcal{C}_i \xrightarrow{\sim} \mathcal{C}_i \times \prod_{j \neq i} \{X_j\} \hookrightarrow \prod \mathcal{C}_j \xrightarrow{F} \mathcal{D}$$

is n_i-excisive.

Similarly for <u>$\vec{n}$-reduced</u> & <u>$\vec{n}$-homogeneous</u>.

F is simply called <u>excisive</u> if it is $(1,\dots,1)$-excisive.

Similarly for <u>reduced</u>.

If F is reduced & excisive, it is called <u>multilinear</u>.

Rmk: F is reduced $\Leftrightarrow$ $\forall \vec{X}$ s.t. one X_i is terminal, $F(\vec{X})=1$.

Let $\mathrm{Exc}^{\vec{n}}(\mathcal{C}_1,\dots,\mathcal{C}_n,\mathcal{D}) \subset \mathrm{Fun}(\prod \mathcal{C}_j,\mathcal{D})$ be the $\vec{n}$-excisive functors.

<u>Rmk</u>

$$\begin{tikzcd}
\mathrm{Exc}^{\vec{n}}(\mathcal{C}_1,\dots,\mathcal{C}_n,\mathcal{D}) \arrow[r, hook, "i_{\vec{n}}"] \arrow[d, "\wr"] & \mathrm{Fun}(\mathcal{C}_1\times\prod_{j>1}\mathcal{C}_j,\mathcal{D}) \arrow[d, "\wr"] \\
\mathrm{Exc}^{n_1}(\mathcal{C}_1,\mathrm{Exc}^{\vec{n}'}(\mathcal{C}_2,\dots,\mathcal{C}_n,\mathcal{D})) \arrow[r, hook] & \mathrm{Fun}(\mathcal{C}_1,\mathrm{Fun}(\prod_{j>1}\mathcal{C}_j,\mathcal{D}))
\end{tikzcd}$$

where $\vec{n}'=(n_2,\dots,n_m)$.

Easy corollary: $i_{\vec{n}}$ admits a left-exact left-adjoint $P_{\vec{n}}$,

defined by induction on m by $P_{\vec{n}} = (P_{\vec{n}'})^* \circ P_{n_1}$.

<u>Prop</u> (7.1.3.4) (Time-permitting will give proof later) $\rightsquigarrow$ If $F:\mathcal{C}^m\to\mathcal{C}$ is $\vec{n}$-exc $\Rightarrow$ $F\circ\Delta$ is n-exc.

Let $\mathcal{C}_1,\dots,\mathcal{C}_m$ have finite colimits, and $\mathcal{D}$ finite limits.

If $F:\prod_j \mathcal{C}_j \longrightarrow \mathcal{D}$ is $\vec{n}$-excisive, then as a functor of one variable, F is $n := \sum n_i$-excisive.

There is also a partial converse:

<u>Prop</u> If additionally, each $\mathcal{C}_i$ is good, and F is reduced (in each variable), and F is m-excisive as a functor of one-variable, then it is $(1,\dots,1)$-excisive.

(7.1.3.13)

<u>Cor</u>: (7.1.3.14)

Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be good, $\mathcal{D}$ diff'l, and $F : \prod_j \mathcal{C}_j \to \mathcal{D}$ reduced (in each variable). Then

$$\underset{\substack{\uparrow \\ \text{functor of one-variable}}}{P_m}(F) \simeq P_{1,\ldots,1}(F).$$

<u>Pf</u>: $P_{1,\ldots,1}(F)$ is m-excisive, so the unit map

$$F \to P_{1,\ldots,1}(F) \text{ factors uniquely as}$$

$$F \to P_m(F) \xrightarrow{\alpha} P_{1,\ldots,1}(F).$$

Fix $1 \leq i \leq m$ and let $\mathcal{E}_i \subset \prod_j \mathcal{C}_j$ be spanned by $\vec{X}$ s.t. $X_i \simeq 1_i$. $j_i : \mathcal{E}_i \hookrightarrow \prod_j \mathcal{C}_j$ preserves pushouts + terminal objects

$\Rightarrow \quad P_m(j_i^* F) \simeq j_i^*(P_m(F))$. (read $P_m(F \circ j_i) \simeq P_m(F) \circ j_i$)

F is reduced $\Rightarrow j_i^* F$ is terminal $\Rightarrow$ so is $P_m(j_i^* F)$.

$\Rightarrow P_m(F)$ is reduced & m-excisive, $\therefore$ $(1, \ldots, 1)$-excisive.

So the unit map $F \to P_m(F)$ factors uniquely as

$$F \to P_{1,\ldots,1}(F) \xrightarrow{\beta} P_m(F).$$

By uniqueness, α & β are homotopy inverses.

<u>Prop</u>: Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be good & $\mathcal{D}$ diff'l, and $F : \prod_j \mathcal{C}_j \to \mathcal{D}$ $(1, \ldots, 1)$-reduced. Then F is m-reduced.

(7.1.3.10).

Construction (7.1.3.15):

Suppose $\mathcal{C}_1, \dots, \mathcal{C}_m$ have terminal objects & let $\mathcal{D}$ be ptd & have finite limits. Let $S = \overset{\{1,\dots,m\}}{[m]}$ and define $\alpha : \prod_i \mathcal{C}_i \times P(S) \longrightarrow \prod_i \mathcal{C}_i$

by $\alpha(\vec{X}, T)_i = \begin{cases} X_i & \text{if } i \notin T \\ 1_i & \text{if } i \in T. \end{cases}$

If $T' \subset T$: • if $i \in T'$ or $i \notin T$, then $\alpha(\vec{X}, T')_i = \alpha(\vec{X}, T)_i$

• if $i \in T \setminus T'$, $\alpha(\vec{X}, T')_i = X_i \longrightarrow 1_i = \alpha(\vec{X}, T)_i$

Let $F : \mathcal{C}_1 \times \dots \times \mathcal{C}_m \longrightarrow \mathcal{D}$ and ~~let $\tilde{F} := F \circ \alpha$ and~~

$F^T := F \circ \alpha(\bullet, T)$.

Define $\mathrm{Red}(F) := \mathrm{fib}\big(F = F^{\varnothing} \longrightarrow \varprojlim_{\varnothing \neq T \subseteq S} F^T\big)$.

$\mathrm{Red}(F)$ is called the <u>reduction</u> of F.

<u>Prop</u> $\mathrm{Red}(F)$ is reduced & the functor (Prop 7.1.3.17 & Cor 18)

$\mathrm{Red} : \mathrm{Fun}(\prod_i \mathcal{C}_i, \mathcal{D}) \longrightarrow \mathrm{Fun}_*(\prod_i \mathcal{C}_i, \mathcal{D})$ is right adjoint

to the inclusion.

<u>Pf</u> $\mathrm{Red}(F)$ is reduced:

WTS $\forall \vec{X}$ s.t. $X_j = 1_j$ for some j, $\mathrm{Red}(F)(\vec{X}) = 0$.

<u>Note</u>: $\forall T \subset S$, the canonical map

(*) $F^T(\vec{X}) \xrightarrow{\sim} F^{T \cup \{j\}}(\vec{X})$ is an eq'l.

Let $P_{\{j\}}(S) = \{ \{j\} \subseteq T \subseteq S \} \underset{r}{\overset{l}{\rightleftarrows}} P_{\varnothing}(S)$

$r(T) = T \cup \{j\}$, $l \dashv r$

Let $F\alpha_{\vec{X}} = F \circ \alpha(\bullet, \vec{X})$

$$\Rightarrow \boxed{\lim_{\substack{\longleftarrow \\ P_0(S)}} r^*(\ell^* F\alpha_{\vec{x}}) \simeq \lim_{\substack{\longleftarrow \\ P_{\{j\}}(S)}} \ell^* F\alpha_{\vec{x}}} \qquad (\ell + r)$$

$$(*) \quad \begin{array}{c} \text{IS} \\ \lim_{\substack{\longleftarrow \\ \phi \neq T \subseteq S}} F^T \end{array} \qquad \begin{array}{c} \text{IS} \\ (\mathrm{Ran}_{id}\, \ell^* F\alpha_{\vec{x}})(\{j\}) \\ \text{IS} \\ (\ell^* F\alpha_{\vec{x}})(\{j\}) \simeq F^{\{j\}}(\vec{X}) \end{array}$$

$$\Rightarrow \mathrm{Red}(F)(\vec{X}) = \mathrm{fib}(\underbrace{F^{\phi}(\vec{X}) \longrightarrow F^{\{j\}}(\vec{X})}_{\text{eq'l by } (*)}) = 0.$$

Prove the rest if time.

Rmk: (7.1.3.19)

Suppose $\mathcal{C}_1, \ldots, \mathcal{C}_m$ are good and $\mathcal{D}$ is pointed. Let $\mathcal{C} = \prod_j \mathcal{C}_j$.

$\forall n$, $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are left-exact $\Rightarrow$

$\forall\, F : \mathcal{C} \longrightarrow \mathcal{D}$

$P_n(\mathrm{Red}(F)) \simeq \mathrm{Red}(P_n(F))$.

Assume now $n = m$, then $\mathrm{Red}(F)$ reduced $\Rightarrow$

$P_{(1,\ldots,1)}(\mathrm{Red}(F)) \simeq P_m(\mathrm{Red}(F)) \simeq \mathrm{Red}(P_m(F))$.

Construction (7.1.3.20):

Let $\mathcal{C}$ be good & $\mathcal{D}$ be ptd & w/ finite limits.

Consider
$$q : \mathcal{C}^n \longrightarrow \mathcal{C}$$
$$\vec{X} \longmapsto \coprod_i X_i$$

$\forall\, F : \mathcal{C} \longrightarrow \mathcal{D}$, define $\mathrm{cr}_n(F) := \mathrm{Red}(F \circ q)$.

$\mathrm{cr}_n(F)$ is the <u>n^{th}-cross effect of F.</u>

<u>Prop</u>: Let $\mathcal{C}$ be good & let $\mathcal{D}$ be ptd & diff'l, and let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be n-excisive. Then $\forall\ m \leq n+1$, the cross-effect $cr_m(F): \mathcal{C}^m \longrightarrow \mathcal{D}$ is

$(n-m+1, \ldots, n-m+1)$-excisive.

<u>Cor</u> F as above $\Rightarrow cr_{n+1}(F)$ is terminal.

<u>Rmk</u> (7.1.3.23)

Let $\mathcal{C}$ be good & $\mathcal{D}$ ptd & diff'l, and $F: \mathcal{C} \to \mathcal{D}$.

$$\begin{aligned} P_{(1,\ldots,1)}(cr_n(F)) &= P_{(1\ldots 1)}(\mathrm{Red}(F \circ q)) \\ &\simeq \mathrm{Red}(P_n(F \circ q)) \\ &\simeq \mathrm{Red}(P_n(F) \circ q) = cr_n(P_n(F)). \end{aligned}$$

Note cr_n is left-exact since q^* is $\Rightarrow$

$$\begin{aligned} cr_n(D_n(F)) &= cr_n(\mathrm{fib}(P_n(F) \longrightarrow P_{n-1}(F))) \\ &\simeq \mathrm{fib}(cr_n(P_n(F)) \longrightarrow \underbrace{cr_n(P_{n-1}(F))}_{\text{terminal}}) \\ &\simeq cr_n(P_n(F)) \simeq P_{(1,\ldots,1)}(cr_n(F)). \end{aligned}$$

Symmetric Functors.

Recall that for any (discrete) group G, we have

$$EG : \overset{\to cat}{BG} \longrightarrow Set^{\Delta^{op}}$$

with $\varinjlim EG = N(BG)$ $(= BG)$

Given $K \in Set^{\Delta^{op}}$, the canonical $\Sigma_n \circlearrowright K^n \longleftrightarrow \widetilde{K}^n : B\Sigma_n \longrightarrow Set^{\Delta^{op}}$,

Define $K^{(n)} := K^n \times E\Sigma_n / \Sigma_n = \widetilde{K}^n \underset{\Sigma^n}{\otimes} E\Sigma_n \in Set^{\Delta^{op}}$.

This is a homotopy colimit in $(Set^{\Delta^{op}}, \text{Joyal})$ & since $id : (Set^{\Delta^{op}}, \text{Joyal}) \longrightarrow (Set^{\Delta^{op}}, \text{Quillen})$ is left-Quillen, also in $(Set^{\Delta^{op}}, \text{Quillen})$.

If $\mathcal{C}$ is an ∞-cat, so is $\mathcal{C}^{(n)}$, and if $\widetilde{\mathcal{C}}^n : B\Sigma_n \to \widehat{Cat}_\infty$, encodes $\Sigma_n \overset{\rho}{\circlearrowright} \mathcal{C}^n$, $\mathcal{C}^{(n)} \simeq \varinjlim \widetilde{\mathcal{C}}^n \Rightarrow B\Sigma_n \times \mathcal{C}^n \underset{\rho}{\overset{pr}{\rightrightarrows}} \mathcal{C}^n \xrightarrow{\theta} \mathcal{C}^{(n)}$ is a colimit.

<u>Def</u> $\mathcal{C}^{(n)}$ is the <u>n^{th}-extended power</u> of $\mathcal{C}$. If $\mathcal{D}$ is another ∞-cat, a <u>symmetric n-ary functor</u> from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{(n)} \longrightarrow \mathcal{D}$. Denote this ∞-cat by

$$SymFun^n(\mathcal{C}, \mathcal{D}) := Fun(\mathcal{C}^{(n)}, \mathcal{D}).$$

<u>Note</u>

$$\begin{aligned}
Fun(\mathcal{C}^{(n)}, \mathcal{D}) &\simeq Hom_{Cat_\infty}\big(\varinjlim (B\Sigma_n \times \mathcal{C}^n \underset{\rho}{\overset{pr}{\rightrightarrows}} \mathcal{C}^n), \mathcal{D}\big) \\
&\simeq \varprojlim \big(Hom_{Cat_\infty}(B\Sigma_n \times \mathcal{C}^n, \mathcal{D}) \underset{\rho^*}{\overset{pr^*}{\leftleftarrows}} Hom_{Cat_\infty}(\mathcal{C}^n, \mathcal{D})\big) \\
&\simeq \varprojlim \big(Fun(B\Sigma_n, Fun(\mathcal{C}^n, \mathcal{D})) \leftleftarrows Fun(\mathcal{C}^n, \mathcal{D})\big) \\
&\simeq Fun(\mathcal{C}^n, \mathcal{D})^{\Sigma_n} = \text{homotopy invariants}
\end{aligned}$$

So a functor $F : \mathcal{C}^{(n)} \longrightarrow \mathcal{D}$ is the same as a functor $G : \mathcal{C}^n \longrightarrow \mathcal{D}$ equipped with equivalences $(G \simeq \theta^* F)$

$$G(\pi) : G(X_1, \dots, X_n) \xrightarrow{\sim} G(X_{\pi(1)}, \dots, X_{\pi(n)}) \quad \forall\, \pi \in \Sigma_n$$

+ homotopy coherent equivalences

$$G(\sigma\circ\pi) \longrightarrow G(\pi)\circ G(\sigma).$$

Example: If $\mathcal{C}$ is symmetric monoidal, then

$$\begin{aligned}\mathcal{C}^n &\longrightarrow \mathcal{C}\\ (X_1,\dots,X_n) &\longmapsto X_1\otimes\cdots\otimes X_n\end{aligned}$$

extends to a symmetric n-ary functor in a canonical way.

A symmetric n-ary functor $F:\mathcal{C}^{(n)}\to\mathcal{D}$ is reduced if θ^*F is. Denote this ∞-cat by $\mathrm{SymFun}^n_*(\mathcal{C},\mathcal{D})$

" " multilinear " "

" " $\mathrm{SymFun}^n_{lin}(\mathcal{C},\mathcal{D})$.

Rmk

$$\begin{tikzcd}
\mathrm{SymFun}^n_*(\mathcal{C},\mathcal{D}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, "\theta^*"'] & \mathrm{SymFun}^n(\mathcal{C},\mathcal{D}) \arrow[l, shift left, "\exists\, \mathrm{Red}_{(sym)}"] \arrow[d, "\theta^*"] \\
\mathrm{Fun}_*(\mathcal{C}^n,\mathcal{D}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{C}^n,\mathcal{D}) \arrow[l, shift left, "\mathrm{Red}"]
\end{tikzcd}$$

Def: (Assume $\mathcal{C}$ has finite coproducts and a terminal object & $\mathcal{D}$ is ptd & has finite limits). Then $(\mathcal{C},\amalg)$ is symmetric monoidal $\Rightarrow$ $\amalg:\mathcal{C}^{(n)}\longrightarrow\mathcal{C}$ is symmetric.

The n^{th}-symmetric cross effect of F is

$$cr_{(n)}(F) := \mathrm{Red}_{(sym)}(F\circ\amalg)\in\mathrm{SymFun}^n_*(\mathcal{C},\mathcal{D})$$

Note By the rmk, $\theta^* cr_{(n)}(F)\simeq cr_n(F)$

Main Thm:

Let $\mathcal{C}$ be a pointed good ∞-category and $\mathcal{D}$ a pointed differentiable ∞-category. Then symmetric-cross effects induce a functor

$$cr_{(n)} : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}),$$

$cr_{(n)}$ is full & faithful and its essential image is $\mathrm{SymFun}^n_{lin}(\mathcal{C}, \mathcal{D})$.

To prove this we need

Lemma: In the situation above, $cr_{(n)}$ is conservative (i.e. reflects equivalences). (7.1.4.10).

Lets assume this.

Pf of thm:

Let $\amalg : \mathcal{C}^{(n)} \longrightarrow \mathcal{C}$. One can show that $\amalg$ admits left Kan extensions so the functor

$$\amalg^* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}^{(n)}, \mathcal{D}) = \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$$

has a left-adjoint $\amalg_! : \varphi \longmapsto \mathrm{Lan}_{\amalg}\varphi$.

Concretely, given $\varphi \in \mathrm{Fun}(\mathcal{C}^{(n)}, \mathcal{D})$, we have $\rho^*\theta^*\varphi \simeq pr^*\theta^*\varphi$
$\left(B\Sigma_n \times \mathcal{C}^n \overset{\rho \to \text{action}}{\underset{pr}{\rightrightarrows}} \mathcal{C}^n \right)$ The composite $\mathcal{C} \times B\Sigma_n \xrightarrow{\Delta \times id} \mathcal{C}^n \times B\Sigma_n \xrightarrow{\rho^*\theta^*\varphi} \mathcal{D}$

$$\longleftrightarrow \quad B\Sigma_n \xrightarrow{X_\varphi} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad \amalg_!(\varphi) \simeq \varinjlim X_\varphi.$$

$$* \longmapsto \theta^*\varphi \circ \Delta \qquad\qquad \overset{\shortparallel}{\vec{X} \mapsto \varphi(\vec{X})_{\Sigma_n}}$$

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C},\mathcal{D}) \arrow[r, shift right, "\amalg^*"'] & \mathrm{Fun}(\mathcal{C}^{(n)},\mathcal{D}) \arrow[l, shift right, "\amalg_!"'] \arrow[l, phantom, "\perp"] \arrow[r, shift right, "\mathrm{Red}_{(sym)}"'] & \mathrm{Fun}_*(\mathcal{C}^{(n)},\mathcal{D}) = \mathrm{SymFun}^n_*(\mathcal{C},\mathcal{D}) \arrow[l, hook', shift right, "i"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

$\Rightarrow \; i^*\amalg_! := L \dashv \mathrm{Red}_{(sym)} \circ \amalg^* = cr_{(n)}$.

Now suppose $F : \mathcal{C} \to \mathcal{D}$ is n-homogeneous.
Then $\theta^* cr_{(n)} F = cr_n F = \mathrm{Red}(\amalg^* F)$ is $(1,\dots,1)$-reduced,
and we also know F n-excisive $\Rightarrow cr_n(F)$ is $(1,\dots,1)$-excisive
$\Rightarrow cr_{(n)}(F) \in \mathrm{SymFun}^n_{lin}(\mathcal{C},\mathcal{D})$.

<u>Observation</u>: Suffices to assume $\mathcal{D}$ is stable since:

$$\begin{tikzcd} Sp(\mathrm{Homog}^n(\mathcal{C},\mathcal{D})) \simeq \mathrm{Homog}^n(\mathcal{C}, Sp(\mathcal{D})) \arrow[r, "cr_{(n)}"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{SymFun}^n_{lin}(\mathcal{C}, Sp(\mathcal{D})) \arrow[d, "\wr"] \\ \mathrm{Homog}^n(\mathcal{C},\mathcal{D}) \arrow[r, "cr_{(n)}"] & \mathrm{SymFun}^n_{lin}(\mathcal{C},\mathcal{D}) \end{tikzcd}$$

$\leftarrow$ similarly

Now assume $\mathcal{D}$ is stable: $\mathrm{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{P_n} \mathrm{Exc}^n(\mathcal{C},\mathcal{D}) \overset{j}{\hookrightarrow} \mathrm{Fun}(\mathcal{C},\mathcal{D})$
$\Rightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$ is stable, $\Rightarrow$
is right-exact. $j \circ P_n$ preserves seq. colims $\Rightarrow$ preserves all countable colimits.
Also, n-reduced functors are stable under countable colimits
$\Rightarrow \mathrm{Homog}^n(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{D})$ is stable under countable colimits.

Now let $\varphi \in \mathrm{SymFun}^n_{lin}(\mathcal{C},\mathcal{D}) \Rightarrow \theta^*\varphi$ is $(1,\dots,1)$-homogeneous
$\Rightarrow \theta^*\varphi \circ \Delta : \mathcal{C} \to \mathcal{C}^n \to \mathcal{D}$ is n-homogeneous
is $X_\varphi(*)$, $* \in B\Sigma_n$ is the base point.
$\Rightarrow L(\varphi) = \varinjlim X_\varphi$ is n-homogeneous.

So by abuse of notation we have an adjunction

$$\mathrm{Homog}^n(\mathcal{C},\mathcal{D}) \underset{cr_{(n)}}{\overset{L}{\rightleftarrows}} \mathrm{SymFun}^{n}_{lin}(\mathcal{C},\mathcal{D}).$$

To show it's an eq'l, it suffices to show that the unit & co-unit are eq'ls.

Suppose for a moment that one can show the unit

$\eta: id \longrightarrow cr_{(n)}\circ L$ is an eq'l.

Since $L \dashv cr_{(n)} \Rightarrow \forall F \in \mathrm{Homog}^n(\mathcal{C},\mathcal{D})$,

$$cr_{(n)}(\varepsilon_F)\circ \eta_{cr_{(n)}F} \simeq id_{cr_{(n)}F}, \text{ where } \varepsilon = \text{co-unit}$$

$\Rightarrow cr_{(n)}(\varepsilon_F)$ is an eq'l, but $cr_{(n)}$ is conservative $\Rightarrow$

ε_F is an eq'l.

So it suffices to show that η is an eq'l.

<u>Sketch</u>: Suffices to show $\forall \varphi$ that

$$\Theta^*\varphi \xrightarrow{\Theta^*\eta} \Theta^* cr_{(n)} L\varphi \simeq cr_n L\varphi \text{ is an eq'l.}$$

$\mathcal{D}$ is stable $\Rightarrow$ colimits commute w/ fin. limits and cr_n is constructed out of finite limits $\Rightarrow$

$$cr_n(\varinjlim X_\varphi) \simeq \varinjlim cr_n\circ X_\varphi$$

$$B\Sigma_n \xrightarrow{X_\varphi} \mathrm{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{cr_n} \mathrm{Fun}_*(\mathcal{C}^n,\mathcal{D})$$

$$* \longmapsto cr_n(\Theta^*\varphi\circ\Delta)$$

is $\leftarrow$ <u>Lemma</u> 7.1.4.13

$$\bigoplus_{\sigma\in\Sigma_n}\Theta^*\varphi\circ\tilde{\sigma}, \quad \tilde{\sigma}: \mathcal{C}^n\to\mathcal{C}^n$$

$\Rightarrow \Theta^*\varphi \overset{\exists}{\xrightarrow{\sim}} \Theta^* cr_{(n)} L\varphi$, and more carefully, one can show this map is $\Theta^*\eta$.

Extra Time: $\mathcal{C}_1 \ldots \mathcal{C}_m$ have $\mathbb{1}$, $\mathcal{D}$ ptd w/ fin limits

Let's show $\mathrm{Red}: \mathrm{Fun}(\prod_i \mathcal{C}_i, \mathcal{D}) \longrightarrow \mathrm{Fun}_*(\prod \mathcal{C}_i, \mathcal{D})$

is right-adjoint to the inclusion. Let $S = \{1, \ldots, m\}$

Let $G: \mathcal{C}_1 \times \cdots \times \mathcal{C}_m \longrightarrow \mathcal{D}$ be reduced.

Recall, $\mathrm{Red}(F) \simeq \mathrm{fib}(F = F^{\emptyset} \longrightarrow \varprojlim_{\emptyset \neq T \subseteq S} F^T) \Rightarrow$

$$\mathrm{Hom}(G, \mathrm{Red}(F)) \xrightarrow{\beta} \mathrm{Hom}(G, F) \longrightarrow \varprojlim_{\emptyset \neq T \subseteq S} \mathrm{Hom}(G, F^T)$$

is a fiber sequence of spaces. WTS β is an equiv.

Suffices to show that each $\mathrm{Hom}(G, F^T)$ is contractible for $T \neq \emptyset$.

Fix $j \in T$, and let $\mathcal{E} \overset{\ell}{\hookrightarrow} \mathcal{C}_1 \times \cdots \times \mathcal{C}_m$ be s.t. $\vec{X} \in \mathcal{E} \Leftrightarrow X_j = \mathbb{1}_j$.

Note: ℓ admits $\alpha(\cdot, \{j\})$ as a right-adjoint $\Rightarrow$

$$F^T \simeq \mathrm{Ran}_\ell\, \ell^* F^T$$

$$\Rightarrow \mathrm{Hom}(G, F^T) \simeq \mathrm{Hom}(G, \mathrm{Ran}_\ell\, \ell^* F^T) \simeq \mathrm{Hom}(\ell^* G, \ell^* F^T).$$

But G is reduced $\Rightarrow$ $\ell^* G$ is terminal & $\therefore$ initial since

$\mathcal{D}$ is pointed, $\therefore$ $\mathrm{Hom}(G, F^T) \simeq *$. $\square$