GENERALIZED WITT SCHEMES IN ALGEBRAIC TOPOLOGY

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Abstract. We analyze the even-periodic cohomology of the space BU and some of its relatives using the language of formal schemes as developed by Strickland. In particular, we connect $E^0(BU)$ to the theory of Witt vectors and λ-rings. We use these connections to study the effect of the coproduct arising from the tensor product on generalized Chern classes. We then exploit this connection to simultaneously construct Husemoller's splitting of $H\mathbb{Z}^*_{(p)}(BU)$ and Quillen's splitting of $MU_{(p)}$.

1. Introduction

> *It is also very tempting to declare that at this date all such results on ordinary homology [of BU and BSU] may be assumed known; and if they are not on record, why, that is a defect in the papers written ten or twenty years ago, and not in the present one. Unfortunately, a sense of duty impels me to sketch a proof.*
>
> \- J.F. Adams 1976

This paper picks up threads left by Ben-Zvi [BZ95] and Strickland [Str00b] and weaves them together. In Ben-Zvi's minor thesis he observes that the cohomology ring $H^*(BU)$ is the ring of functions on the Cartier dual of the Witt scheme. While Strickland is developing the proper foundations for making these kind of connections in [Str00b] he also remarks that the formal scheme associated to $E^0(\coprod_{n\geq 0} BU(n))$, for E an even-periodic cohomology theory, is a graded λ-semiring (or rig) object. Using Strickland's framework to analyze the E cohomology of a space by studying the associated formal scheme $\mathrm{Spf}(E^0(BU))$ and the algebra of Witt schemes described in [BZ95, Haz78], we will flesh out and connect these remarks.

This continues a tradition initiated by Morava in the 70's, a strong advocate of applying the language and tools of algebraic geometry to the study of algebraic topology. Some of his ideas eventually flowered into the field of derived algebraic geometry. In this paper, we will study some of the ordinary algebraic geometry that motivated the creation of this blossoming field.

Witt vectors appear in many places in algebraic topology including the Husemoller-Witt splitting [Hus71], Hodgkin's calculation of the K-theory of QS^0 [Hod72], the classification of bicommutative Hopf algebras [Sch70, Goe98] and in the formulas for formal group laws [Rav00].

Individually the results below are in the literature although they may require some translation since they appear in different contexts. For example, our form of the algebraic splitting principle Theorem 3.9, can be constructed from different forms of the splitting principle that appear in the literature. We have made a significant effort to cite relevant sources, but some of the topics

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discussed below have been studied by many people for quite some time. By juxtaposing the results spread out across the literature we aim to clarify the connections between them.

This paper is written for the general algebraic topologist. We therefore assume familiarity with some category theory, especially representable functors and the Yoneda lemma as well as the theory of vector bundles and classifying spaces laid out in [Hus94, May99]. But we do not assume the reader is familiar with the basics of affine schemes or ind-objects. To keep this paper mostly self-contained, in Section 2 we provide a short treatment of these topics referring to [Str00b] for a more in-depth treatment.

First, in Section 2.1 we recall the relevant material about schemes, while emphasizing the role of the representing algebra. After reviewing the fundamentals of ind and pro-objects in Section 2.2 we then proceed to the theory of formal schemes in Section 2.3 and Cartier duality in Section 2.4.

In Section 3 we introduce the star players for the algebraic geometry team. We introduce our theory of symmetric schemes which provide a simple intermediary between Λ-schemes and Witt schemes. After recalling some aspects of the theory of λ-rings and Witt vectors, including their p-local splittings, (see [Haz78, BZ95] or the recent survey [Haz08]) we explicitly describe the equivalences between them. Symmetric schemes encode the splitting principle rather explicitly, allowing us to easily define operations on symmetric schemes.

In Section 4 we introduce the players for the algebraic topology team. We identify the schemes associated to the even-periodic co/homology of spaces related to BU with some of the schemes above and apply our algebraic results. In particular, we use the p-local splitting of the Witt scheme to construct the Husemoller splitting of the cohomology of BU and Quillen's splitting of $MU_{(p)}$ simultaneously. The observation that these splittings can be constructed simultaneously appears to be new.

We also apply this framework to give low-dimensional formulas—which although well-known to some experts, apparently have never been published—for the generalized Chern class of a tensor product of (stable) vector bundles in Section 5.

1.1. **Conventions.** In this paper:

- A ring will always be commutative, associative, and unital.
- A ring without identity will be called a rng.
- A ring without negatives (additive inverses) will be called a rig.
- All binary operations considered will be associative and dually all co-operations will be coassociative.
- All schemes and formal schemes will be affine.

2. Affine algebraic geometry

2.1. **Schemes.** We are interested in studying the cohomology of suitably nice spaces that admit a homotopy commutative and unital product. The cohomology rings of such spaces come equipped with a cocommutative and counital coproduct. Being more comfortable with product structures we choose to work in the opposite category of rings or more precisely, the category of affine schemes. In addition to easing our study of the comultiplicative structure on these cohomology rings, schemes provide interesting alternative characterizations of these rings.

Recall that the essential image of an embedding $F : \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ whose objects are isomorphic to some object in the image of F.

Definition 2.1. *The category of affine schemes, $\mathcal{Sch}$, is defined to be the essential image of the Yoneda embedding:*

$$\begin{array}{rcl} \mathrm{Spec} : \mathcal{Ring}^{op} & \longrightarrow & \mathcal{Set}^{\mathcal{Ring}} \\ \mathrm{Spec}(R) : S & \mapsto & \mathcal{Ring}(R, S). \end{array}$$

The value of a scheme X on a ring S, is called the set of S-points of X which we denote by $X(S) = \mathcal{Sch}(\mathrm{Spec}(S), X)$.

By definition, $\mathcal{Sch}$ is equivalent to $\mathcal{Ring}^{op}$.

Example 2.2. The affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$, takes a ring to its underlying set. In other words, $\mathbb{A}^1$ is isomorphic to the forgetful functor from rings to sets.

Example 2.3. The scheme $\mathbb{A}^1 \setminus \{*\} \cong \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$, takes a ring to the set of units in that ring.

Example 2.4. The scheme $\mathrm{Nil}_n \cong \mathrm{Spec}(\mathbb{Z}[x]/(x^n))$, takes a ring R to the set of x in R, such that $x^n = 0$.

To discuss various algebraic categories in $\mathcal{Sch}$, such as group schemes, we will need finite products. The product in schemes arises from the tensor product in rings. As an example we consider affine n-space

$$\begin{array}{rcl} \mathbb{A}^n & \cong & \left(\mathbb{A}^1\right)^{\times n} \\ & = & \mathrm{Spec}\,\mathbb{Z}[t_1] \times \cdots \times \mathrm{Spec}\,\mathbb{Z}[t_n] \\ & \cong & \mathrm{Spec}(\mathbb{Z}[t_1] \otimes \cdots \otimes \mathbb{Z}[t_n]) \\ & \cong & \mathrm{Spec}\,\mathbb{Z}[t_1, \ldots, t_n]. \end{array}$$

Remark 2.5. In fact, $\mathcal{Sch}$ is complete and cocomplete because $\mathcal{Ring}$ is complete and cocomplete, however we must comment that the colimits in affine schemes constructed using this equivalence do not generally agree with those of the larger categories of non-affine schemes or $\mathcal{Set}^{\mathcal{Ring}}$. Since the reader might have little intuition for schemes, we will try to emphasize their role as set-valued functors and clarify the differences between these perspectives.

Later we will need to work with k-algebras, for an arbitrary ring k, whose scheme theoretic analogues are schemes over $\mathrm{Spec}(k)$. The category of schemes over $\mathrm{Spec}(k)$ is equivalent to the essential image of the Yoneda embedding:

$$\mathrm{Spec}_k : k\text{-}\mathcal{alg}^{op} \longrightarrow \mathcal{Set}^{k\text{-}\mathcal{alg}},$$

where k-$\mathcal{alg}$ is the category of k-algebras (i.e., rings under k). We denote the category of schemes over $\mathrm{Spec}(k)$ by $\mathcal{Sch}_k$. Note that the isomorphism of categories $\mathcal{Sch} \cong \mathcal{Sch}_{\mathbb{Z}}$, shows that the relative theory is more general.

Remark 2.6. In our examples we have elected to define our schemes over $\mathbb{Z}$, but we could just as easily define their analogues over an arbitrary base ring. Rather than clutter the notation we have elected to leave it to the reader to replace the integers with her preferred base whenever she sees fit.

Tensor products over k correspond to products in $\mathcal{Sch}_k$ and they agree with those in $\mathcal{Set}^{k\text{-}\mathcal{alg}}$. Using this product structure we can construct the category of group schemes over $\mathrm{Spec}\,k$, $\mathcal{GrpSch}_k$. We should remark that group schemes are the dual of a more familiar concept:

Proposition 2.7. *The categories of bicommutative Hopf algebras over k and group schemes over k are antiequivalent.*

Example 2.8. The ring $\mathbb{Z}[x]$ with augmentation $\epsilon_+ : \mathbb{Z}[x] \to \mathbb{Z}$ determined by $\epsilon_+(x) = 0$ is an augmented $\mathbb{Z}$-algebra. The maps

$$\begin{array}{rclcrcl} \epsilon_+ : \mathbb{Z}[x] & \to & \mathbb{Z} & & x & \mapsto & 0 \\ \Delta_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x_1, x_2] & & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ \chi_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x] & & x & \mapsto & -x \end{array}$$

make $\mathbb{Z}[x]$ into a cocommutative cogroup (i.e., a bicommutative Hopf algebra). Applying Spec to $\mathbb{Z}[x]$ and the above maps defines the additive group scheme $\mathbb{G}_a$. We can identify $\mathbb{G}_a$ with the forgetful functor from rings to abelian groups.

Example 2.9. The ring

$$\mathbb{Z}[x^{\pm}] \equiv \mathbb{Z}[x, x^{-1}]$$

with augmentation $\epsilon_\times(x) = 1$ can be made into a cocommutative cogroup using the maps

$$\begin{array}{rclcrcl} \epsilon_\times : \mathbb{Z}[x^{\pm}] & \to & \mathbb{Z} & & x & \mapsto & 1 \\ \Delta_\times : \mathbb{Z}[x^{\pm}] & \to & \mathbb{Z}[x_1^{\pm}, x_2^{\pm}] & & x & \mapsto & x_1 \otimes x_2 \\ \chi_\times : \mathbb{Z}[x^{\pm}] & \to & \mathbb{Z}[x^{\pm}] & & x & \mapsto & x^{-1}. \end{array}$$

The corresponding group scheme $\mathbb{G}_m$ is called the multiplicative group scheme since it takes a ring to its group of units.

Remark 2.10. As an aside, we note that $\mathbb{G}_m$ plays a role in extending the antiequivalence between the categories of rings and affine schemes to graded rings. There is an equivalence between the category of graded algebras and the category of comodules over the Hopf algebra representing $\mathbb{G}_m$ [Str00b, 2.96]. Applying Spec to everything in sight we end up with an antiequivalence between the category of graded algebras and the $\mathbb{G}_m$-equivariant category of affine schemes.

We can combine some of the structure in Example 2.9 with that of Example 2.8 to define the identity ring scheme.

Example 2.11. The ring $\mathbb{Z}[x]$ with augmentations ϵ_+ and $\epsilon_\times$ and comultiplications Δ_+ and $\Delta_\times$ equipped with the coinverse map χ_+ make $\mathbb{Z}[x]$ into a coring. By applying Spec we obtain the ring scheme Id, which takes a ring to itself.

2.2. **Ind-objects and pro-objects.** Before proceeding to the theory of formal schemes, we need to recall some standard facts about ind/pro objects. We encourage the reader to consult [Joh82, Gro64] for a more thorough treatment of this material.

Definition 2.12. *A small category $\mathcal{D}$ is called* cofiltered *if*

(1) *$\mathcal{D}$ is non-empty.*
(2) *For every $X, Y \in \mathcal{D}$, there exists an object $Z \in \mathcal{D}$ and morphisms $f : Z \to X$ and $g : Z \to Y$.*
(3) *For every two arrows $f, g : X \to Y$ there exists an object $Z \in \mathcal{D}$ and a morphism $h : Z \to X$ such that $fh = gh$.*

It follows immediately from the definition that any product of cofiltered categories is cofiltered.

Definition 2.13. *Given a category $\mathcal{C}$ with small hom-sets and a functor $F : \mathcal{D} \to \mathcal{C}$ from a cofiltered category $\mathcal{D}$, we define the ind-object* "colim" $F \in \mathcal{S}et^{\mathcal{C}^{op}}$ *by*

$$\text{“colim”}\, F \in \mathcal{S}et^{\mathcal{C}^{op}} \equiv \operatorname*{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i)),$$

where the colimit is computed in $\mathcal{S}et^{\mathcal{C}^{op}}$.

We compute the morphisms between two ind-objects to be

$$\begin{aligned} \mathcal{S}et^{\mathcal{C}^{op}}(\text{“colim” } F, \text{“colim” } G) &= \mathcal{S}et^{\mathcal{C}^{op}}(\operatorname*{colim}_{i\in\mathcal{D}} \mathcal{C}(-,F(i)), \operatorname*{colim}_{j\in\mathcal{E}} \mathcal{C}(-,G(j))) \\ &\cong \lim_{i\in\mathcal{D}} \mathcal{S}et^{\mathcal{C}^{op}}(\mathcal{C}(-,F(i)), \operatorname*{colim}_{j\in\mathcal{E}} \mathcal{C}(-,G(j))) \\ &\cong \lim_{i\in\mathcal{D}} \operatorname*{colim}_{j\in\mathcal{E}} \mathcal{C}(F(i),G(j))). \end{aligned}$$

The first isomorphism follows from the definition of a colimit and the second by the Yoneda lemma and that colimits in functor categories are computed pointwise.

It follows that the full subcategory of $\mathcal{S}et^{\mathcal{C}^{op}}$ consisting of objects isomorphic to an ind-object has small hom-sets. This is the category $\operatorname{Ind}\mathcal{C}$ of ind-objects in $\mathcal{C}$.

The functor category $\mathcal{S}et^{\mathcal{C}^{op}}$ is complete because $\mathcal{S}et$ is complete. If $\mathcal{C}$ also has finite products then the product

$$\text{“colim” } F \times \text{“colim” } G \in \mathcal{S}et^{\mathcal{C}^{op}}$$

can be realized as an ind-object:

$$\text{“colim” } F \times \text{“colim” } G = \operatorname*{colim}_{i\in\mathcal{D}} \mathcal{C}(-,F(i)) \times \operatorname*{colim}_{j\in\mathcal{E}} \mathcal{C}(-,G(j)) \tag{2.14}$$

$$\cong \operatorname*{colim}_{(i,j)\in\mathcal{D}\times\mathcal{E}} \mathcal{C}(-,F(i)\times G(j)), \tag{2.15}$$

where the isomorphism follows from cofiltered colimits commuting with finite products in $\mathcal{S}et$[Bor94, 2.13.4]. Using this we can relate the algebraic categories in $\mathcal{C}$ to those in $\operatorname{Ind}\mathcal{C}$.

If $\mathcal{C}$ is equivalent to a category of representable functors then we can think of $\operatorname{Ind}\mathcal{C}$ as formally adjoining cofiltered colimits to $\mathcal{C}$:

Theorem 2.16 ([Joh82, Section VI]). *Suppose that* $\mathcal{D}$ *is the subcategory of functors in* $\mathcal{S}et^{\mathcal{C}^{op}}$ *such that for any object* $X \in Obj(\mathcal{D})$, $X \cong \mathcal{C}(-,Y)$ *for some* Y. *Then* $\operatorname{Ind}\mathcal{D}$ *is equivalent to the subcategory* $\mathcal{E}$ *of functors in* $\mathcal{S}et^{\mathcal{C}^{op}}$, *such that for all* $\overline{X} \in Obj(\mathcal{E})$, $\overline{X} = \operatorname{colim} Y_i$. *where* $Y_i \in \mathcal{D}$.

The definition of a pro-object is precisely dual to that of an ind-object.

Notation 2.17. *We denote the category of pro-objects in* $\mathcal{C}$ *by* $\operatorname{Pro}\mathcal{C}$.

Under the tautological equivalence

$$\operatorname{Pro}\mathcal{C} \simeq \operatorname{Ind}\mathcal{C}^{op}, \tag{2.18}$$

we obtain results for pro-objects dual to those above.

2.3. **Formal schemes.** Following [Str00b], we define the category of formal schemes, $\mathcal{FS}ch$, as the full subcategory of objects in $\mathcal{S}et^{\mathcal{R}ing}$ which are isomorphic to a cofiltered colimit of affine schemes. This category is equivalent to the category of ind-schemes. By identifying a scheme with a constant ind-scheme we can embed $\mathcal{S}ch$ as a full subcategory of $\mathcal{FS}ch$.

Using the equivalence with Ind-schemes, $\mathcal{FS}ch$ is equivalent to the opposite category of pro-rings. A pro-ring $R = \text{“lim” } R_i$ can be identified with the topological ring $R' = \lim R_i$ where the inverse limit is taken in topological spaces and each R_i represents a discrete topological space [Joh82]. In this description a map of pro-rings corresponds to a continuous map. When the context is clear we will identify such a topological ring with its associated pro-ring.

The equivalence $\operatorname{Pro}\mathcal{R}ing^{op}$ to $\mathcal{FS}ch$ is given by the functor

$$\operatorname{Spf} : \text{“lim” } R_i \mapsto \operatorname{colim} \operatorname{Spec}(R_i).$$

Remark 2.19. This generalizes the definition of formal schemes in algebraic geometry. An affine formal scheme in that context is one of the form $\mathrm{Spf}(\widehat{R})$, where $\widehat{R} =$ "lim" R/I^n ([Har77, Section II.9]). Each such formal scheme has a geometric interpretation, that does not always hold in our category.

Example 2.20. The formal affine line

$$\begin{aligned}\widehat{\mathbb{A}}^1 &= \mathrm{Spf}(\mathbb{Z}[\![x]\!]) \\ &= \operatorname{colim} \mathrm{Spec}(\mathbb{Z}[x]/x^n) \\ &\cong \operatorname{colim} \mathrm{Nil}_n \\ &= \mathrm{Nil},\end{aligned}$$

takes a ring to the set of nilpotent elements of that ring.

Notation 2.21. *Analogous to the informal case, we have the category of formal schemes over a given (formal) scheme X, which we denote by $\mathcal{FSch}_X$.*

Since colimits commute with (cofiltered) colimits we see that $\mathcal{FSch}$ is cocomplete and its colimits arise from those in schemes. By Theorem 2.16, cofiltered colimits of formal schemes are preserved under the inclusion $\mathcal{FSch} \to \mathcal{Set}^{\mathcal{Ring}}$.

Using Equation 2.15 we see that the products in schemes are related to the products in formal schemes. For example, if $X = \operatorname{colim}_{i\in I} X_i$ and $Y = \operatorname{colim}_{j\in J} Y_j$ then

$$X \times Y \cong \operatorname*{colim}_{(i,j)\in I\times J} X_i \times Y_j.$$

Dually there is a coproduct on pro-rings, called the completed tensor product. As an example, we see that $\widehat{\mathbb{A}}^2$ is represented by

$$\mathbb{Z}[\![x]\!]\widehat{\otimes}\mathbb{Z}[\![y]\!] \cong \mathbb{Z}[\![x, y]\!].$$

Now that we have finite products we can define formal groups, formal rings, etc.

Example 2.22. The additive formal group $\widehat{\mathbb{G}}_a$ takes a ring R to $\mathrm{Nil}^+(R)$, the additive group of nilpotent elements of R. Clearly its underlying formal scheme is isomorphic to $\widehat{\mathbb{A}}^1$. Fixing an isomorphism, then the additive group structure is defined by the following maps:

$$\begin{array}{rcllcl} \epsilon_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z} & x & \mapsto & 0 \\ \Delta_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x_1, x_2]\!] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ \chi_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x]\!] & x & \mapsto & -x. \end{array}$$

Example 2.23. The multiplicative formal group $\widehat{\mathbb{G}}_m$ takes a ring R to the multiplicative group $(1 - \mathrm{Nil}(R))^\times$. Identifying this set with $\mathrm{Nil}(R)$ we see that $\widehat{\mathbb{G}}_m$ is isomorphic to $\widehat{\mathbb{A}}^1$ as a formal scheme. Fixing an isomorphism, the group structure is defined by the following maps:

$$\begin{array}{rcllcl} \epsilon_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z} & x & \mapsto & 0 \\ \Delta_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x_1, x_2]\!] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 - x_1 \otimes x_2 \\ \chi_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x]\!] & x & \mapsto & -\sum_{i\geq 0} x^{i+1}. \end{array}$$

Remark 2.24. Hazewinkel refers to the above formal group law as $\widehat{\mathbb{G}}_m^-$ [Haz78]. The multiplicative formal group is usually defined to represent the multiplicative group $(1 + \mathrm{Nil}(R))^\times$. However, the formal group law in 2.23 naturally occurs as an E_∞ orientation on complex K-theory, while the standard example does not [And95].

2.4. **Cartier duality.** Since our interest is actually in the ring that represents a given scheme, it is desirable to have a theory of duality for schemes that corresponds to taking the linear dual of the representing ring. Of course, for such a duality to exist we are going to need that the dual of the representing ring is another commutative unital ring, which means that the original scheme needs to be a commutative group scheme. In order for a group scheme to be canonically isomorphic to its double dual we are going to need that the representing ring be a dualizable module. Since we also want our theory of duality to apply to formal schemes we are going to need some assumptions about the maps in the pro-systems that define the representing pro-rings. The classical theory of Cartier duality (see [Dem72]), once suitably extended as in [Str00b, Section 6.4], suits our purposes.

Cartier duality is the analogue of Pontryagin duality for group schemes. The classical theory of Cartier duality requires that we work over a field and as a result safely ignores some of the issues described above. When working over more general rings we will need some new algebraic restrictions, to obtain a well-behaved duality theory.

We start by considering a suitable category of objects dual to k-algebras. Identifying k-algebras with the category of commutative monoids in the category of k-modules using the tensor product structure, we see that the appropriate dual is the category of cocommutative comonoids in the category of k-modules or, equivalently, counital, cocommutative coalgebras.

Definition 2.25. *Suppose U is a k-coalgebra free on a basis I. Let $\mathcal{D}_I$ be the category whose objects are the k-subcoalgebras of U which are free modules on a finite subset of I and whose morphisms are inclusions. If there exists a basis I such that*

$$U \cong \operatorname*{colim}_{\mathcal{D}_I} U_i$$

then we say I is a good basis *for U. Those coalgebras which admit a good basis will be called* basic.

Notation 2.26. *We denote the full subcategory of basic coalgebras in the category of coalgebras by $\mathcal{BCoAlg}$.*

Definition 2.27. *To a basic coalgebra $U = \operatorname{colim} U_i$, we define the formal scheme $\operatorname{Sch} U = \operatorname{colim} \operatorname{Spec} U_i^*$, where U_i^* is the linear dual $\operatorname{Mod}_k(U_i, k)$.*

Here $\lim U_i^*$ inherits its multiplicative structure from the coalgebra structure on U_i (see [Str00b, 4.59]). It follows that Sch does indeed define a functor from basic coalgebras to formal schemes whose image $\mathcal{CFSch}$, we call *coalgebraic formal schemes.*

We construct an inverse functor $c : \mathcal{CFSch} \to \mathcal{BCoAlg}$ by setting

$$c(\operatorname{colim} \operatorname{Spec} U_i^*) = \operatorname{colim} U_i^{**} \cong \operatorname{colim} U_i.$$

If our coalgebra U has the additional multiplicative structure making it a commutative Hopf algebra then $\operatorname{Spec} U$ $(\operatorname{Sch} U)$ is a commutative (formal) group scheme. Given a formal coalgebraic group scheme

$$\widehat{\mathbb{G}} \cong \operatorname{colim} \operatorname{Spec} U_i^*,$$

we define the Cartier dual to be

$$\begin{aligned} D\widehat{\mathbb{G}} &= \operatorname{Spec} c\widehat{\mathbb{G}} \\ &\cong \operatorname{Spec} \operatorname{colim} U_i \\ &\cong \operatorname{Spec} U. \end{aligned}$$

Restricting to those coalgebraic formal schemes that are actually informal schemes we can apply D again to get

$$\begin{aligned} DD\widehat{\mathbb{G}} &= D\,\mathrm{Spec}\,c\widehat{\mathbb{G}} \\ &= c\,\mathrm{Spec}\,c\widehat{\mathbb{G}} \\ &\cong c\,\mathrm{Spec}\,U \\ &\cong \mathrm{colim}\,\mathrm{Spec}\,U_i^* \\ &\cong \widehat{\mathbb{G}}. \end{aligned}$$

This gives us a well-behaved duality on group schemes that has the effect of taking the linear dual on the representing rings.

Example 2.28. The linear dual of the truncated polynomial algebra $\mathbb{Z}[x]/(x^n-1)$ with x grouplike (i.e., $\Delta x = x \otimes x$) is the module

$$\bigoplus_{i=0}^{n-1} \mathbb{Z}e_i$$

with coproduct

$$\Delta e_k = \sum_{i=0}^{n-1} e_i \otimes e_{\sigma(i)}$$

where $\sigma(i) \equiv k - i \pmod n$ and $0 \le \sigma(i) < n$. The algebra structure is determined by the relations $e_i e_j = \delta_{ij} e_i$. We can see $\mathrm{Spec}\,\mathbb{Z}[x]/(x^n-1)$ is the group scheme whose R-points are the multiplicative group of nth roots of unity in R (which might be trivial for a given R), while its Cartier dual is the constant functor $R \to \mathbb{Z}/n$.

Remark 2.29. For a coalgebraic commutative formal group scheme $\mathbb{H}$ one can define (see [Str00a, 4.69]) the following *scheme* of maps in commutative formal groups

$$D\mathbb{H} = \mathcal{G}rp\mathcal{S}ch(\mathbb{H}, \mathbb{G}_m).$$

This definition of the functor D most closely resembles the classical definition of Cartier duality and Strickland has shown that these definitions coincide [Str00a, 6.15].

3. Three perspectives

3.1. **Symmetric schemes.** We want to study operations on sets of monic polynomials. Some of these operations are easier to describe under the assumption that the monic polynomials split. As an intermediate step, we examine monic polynomials with a specified splitting, then monic polynomials that have a splitting before proceeding to general monic polynomials.

We will identify a *split* monic polynomial over R

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} b_{n-i} x^i \\ &= \prod_{i=0}^{n} (x - t_i) \end{aligned}$$

with its *unordered* set of roots $\{t_1, \ldots, t_n\}$.

Definition 3.1. *Let the n^{th} splitting functor* Split_n *denote the functor from rings to sets that takes a ring R to the set of split monic polynomials with coefficients in R or, alternatively, the corresponding sets of roots of those polynomials.*

Remark 3.2. Note that Split_n is not a scheme, affine or otherwise, although it is related to a stack.

Definition 3.3. *Let the nth representable splitting functor* rSplit_n *be the affine scheme with the following R-points*

$$\mathrm{rSplit}_n(R) = \left\{ f(x) = \prod_{i=1}^{n}(x - t_i),\ t_i \in R \right\} \cong R^n.$$

Clearly rSplit_n is isomorphic to affine n-space and we have a natural transformation

$$U : \mathrm{rSplit}_n \to \mathrm{Split}_n$$

where we forget the ordering of the roots.

Definition 3.4. *A natural transformation*

$$G : \mathrm{Split}_n \to \mathrm{Split}_m,$$

is algebraic *if there exists a natural transformation $\widetilde{G} : \mathrm{rSplit}_n \to \mathrm{rSplit}_m$ making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{rSplit}_n & \overset{\widetilde{G}}{\dashrightarrow} & \mathrm{rSplit}_m \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Split}_n & \xrightarrow{G} & \mathrm{Split}_n \end{array}$$

Typically, we construct G from $\widetilde{G}$ by checking that $U\widetilde{G}$ factors through U, in which case we abuse notation and write G instead of $\widetilde{G}$ for the algebraic map. For example, we have the map

$$F_k : \mathrm{rSplit}_n \to \mathrm{rSplit}_{nk}$$

sending $(t_1, \ldots, t_n)$ to $([k]t_1, \ldots, [k]t_n)$, where $[k]t_i$ indicates repeat the root t_i k-times. Passing from $([k]t_1, \ldots, [k]t_n)$ to $\{[k]t_1, \ldots, [k]t_n\}$ we see that reordering the t_i's does not change the target set, so the map F_k descends to give an algebraic map. We define this map in terms of its R-points:

$$F_k : \sum_{i=0}^{n} b_{n-i}x^i = \prod_{i=1}^{n}(x - t_i) \mapsto \sum_{i=0}^{nk} b'_{nk-i}x^i = \prod_{i=0}^{n}(x - t_i)^k. \tag{3.5}$$

The coefficient b_i of the polynomial $f(x)$ can be identified with the elementary symmetric polynomial

$$(-1)^{n-i}\sigma_i(t_1, \ldots, t_n)$$

in the roots $t_1, \ldots, t_n$. Here the elementary symmetric functions $\sigma_i(t_1, \ldots, t_n)$ are defined by the following generating function

$$\prod_{i=1}^{n}(x + t_i) = \sum_{i=0}^{n} \sigma_{n-i}x^i.$$

If $i \leq n$ then $\sigma_i(t_1, \ldots, t_n) = \sigma_i(t_1, \ldots, t_n, 0)$, giving us well-defined elementary symmetric functions σ_i on "enough" variables.

Theorem 3.6 (Newton [Hus94])**.** *There is an isomorphism of algebras* $R[\sigma_1, \ldots, \sigma_n] \cong R[t_1, \ldots, t_n]^{\Sigma_n}$.

Since the coefficients b'_i in Equation 3.5 are symmetric in $t_1, \ldots, t_n$ they can be expressed as polynomials in the elementary symmetric functions or, equivalently, as polynomials $p_i(b_1, \ldots, b_n)$ in the coefficients of f. We can now drop the requirement that the polynomial splits and just use the polynomials p_i to define operations on monic polynomials.

Definition 3.7. *Let the nth symmetric scheme* $\mathrm{symm}_n \cong \mathrm{Spec}(\mathbb{Z}[b_1, \ldots, b_n])$ *be the scheme satisfying*

$$\mathrm{symm}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i} x^i \mid b_i \in R, b_0 = 1 \right\}$$

and

We also have formal analogues of the above schemes.

Definition 3.8. *Let the* n^{th} *formal splitting functor* $\widehat{\mathrm{Split}}_n$ *be the formal scheme satisfying*

$$\widehat{\mathrm{Split}}_n(R) = \left\{ f(x) \in \widehat{\mathrm{symm}}_n(R) \mid f(x) = \prod_{i=1}^{n} (x - t_i),\ t_i \in \mathrm{Nil}(R) \right\}.$$

Let the nth formal symmetric scheme $\widehat{\mathrm{symm}}_n \cong \mathrm{Spf}(\mathbb{Z}[\![b_1, \ldots, b_n]\!])$ *be the formal scheme satisfying*

$$\widehat{\mathrm{symm}}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i} x^i \mid b_0 = 1 \text{ and } b_i \in \mathrm{Nil}(R) \text{ for } i > 0 \right\}.$$

Theorem 3.9 (Algebraic Splitting Theorem)**.**

(1) *An algebraic transformation of functors* $\mathrm{Split}_n \to \mathrm{Split}_k$ *determines a map of schemes* $\mathrm{symm}_n \to \mathrm{symm}_k$.
(2) *An algebraic transformation of functors* $\mathrm{Split}_i \times \mathrm{Split}_j \to \mathrm{Split}_k$ *determines a map of schemes* $\mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_k$.
(3) *Two maps* $f_1, f_2 : \mathrm{symm}_m \to \mathrm{symm}_k$ *are equal if and only if* $f_1 U_m = f_2 U_m$ *where* $U_m : \mathrm{rSplit}_m \to \mathrm{symm}_m$ *is the forgetful map.*
(4) *The same results hold for the formal analogues of the above schemes.*

Proof. The argument that F_k induces a natural transformation $\mathrm{symm}_n \to \mathrm{symm}_{nk}$ given above goes through *mutatis mutandis* to prove parts 1 and 2. Namely, in each of these cases we see that the natural transformations on split monic polynomials define polynomial maps on the coefficients which allow us to define natural transformations on monic polynomials.

Part 3 follows from the fact that U_m corresponds to the following *injective* map on representing rings

$$\begin{array}{rcl} \mathbb{Z}[b_1, \ldots, b_m] & \to & \mathbb{Z}[t_1, \ldots, t_m] \\ b_i & \mapsto & \sigma_i(t_1, \ldots, t_m). \end{array}$$

□

The last claim allows us to deduce relations between maps by checking them on the representable (formal) splitting functors.

This allows us to define a panoply of natural transformations. Unless we say otherwise, for each of the following definitions there is an analogous version for the corresponding formal scheme.

Definition 3.10. *Applying Theorem 3.9, let* $\oplus_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_{i+j}$ *be the algebraic map corresponding to*

$$\begin{array}{rcl} \oplus_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j & \longrightarrow & \mathrm{Split}_{i+j} \\ S \times T & \mapsto & S \coprod T. \end{array}$$

In terms of R-points we have another simple description

$$\begin{array}{rcl} \oplus_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j & \to & \mathrm{symm}_{i+j} \\ (f(x), g(x)) & \mapsto & f(x)g(x). \end{array}$$

Construction 3.11. *Given an operation* $\mu : \mathrm{Split}_1 \times \mathrm{Split}_1 \to \mathrm{Split}_1$ *(or equivalently* $\mathrm{symm}_1 \times \mathrm{symm}_1 \to \mathrm{symm}_1$*) we define operations* $\mu_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_{ij}$ *determined by*

$$\begin{array}{rcl} \mu_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j & \to & \mathrm{Split}_{ij} \\ S \times T & \mapsto & \coprod_{(s,t) \in S \times T} \mu(s,t). \end{array}$$

When $\mu(r,t) = rt$ *we will denote the operation* $\mu_{i,j}$ *by* $\otimes_{i,j}$.

Proposition 3.12. *The operations* $\mu_{i,j}$ *defined above Construction 3.11 distribute over* $\oplus_{i,j}$.

Proof. This follows immediately from the definition of $\mu_{i,j}$ and Theorem 3.9. □

Definition 3.13. *Let the map* $i_0 : \mathrm{symm}_0 \longrightarrow \mathrm{symm}_1$, *or equivalently* ($i_0 : \mathrm{Split}_0 \to \mathrm{Split}_1$), *satisfying* $* \mapsto \{0\}$. *Similarly we have a map* $i_1 : \mathrm{symm}_0 \to \mathrm{symm}_1$ *(but not a map* $\widehat{\mathrm{symm}}_0 \to \widehat{\mathrm{symm}}_1$*) defined by* $* \mapsto \{1\}$.

For each n we have an inclusion $\iota : \widehat{\mathrm{symm}}_n \to \widehat{\mathrm{symm}}_{n+1}$ defined as the composite

$$\widehat{\mathrm{symm}}_n \cong \widehat{\mathrm{symm}}_n \times \widehat{\mathrm{symm}}_0 \xrightarrow{id \times i_0} \widehat{\mathrm{symm}}_n \times \widehat{\mathrm{symm}}_1 \xrightarrow{\oplus_{n,1}} \widehat{\mathrm{symm}}_{n+1}.$$

This map takes the monic polynomial $f(x)$ to $x \cdot f(x)$. Now taking a colimit over these maps inverts x and since the colimit of formal schemes agrees with the colimit in $\mathcal{Set}^{\mathcal{Ring}}$ setting $z = x^{-1}$ we obtain:

Definition 3.14. *The formal scheme* $\widehat{\mathrm{symm}}^0 = \mathrm{colim}\, \widehat{\mathrm{symm}}_n \cong \mathrm{Spf}\, \mathbb{Z}[\![b_1, b_2, \ldots]\!]$, *satisfies*

$$\widehat{\mathrm{symm}}^0(R) = \left\{ f(z) = \sum_{i=0}^{n} b_i z^i \mid b_0 = 1,\ n \in \mathbb{N},\ b_i \in \mathrm{Nil}(R) \text{ for } i > 0 \right\}.$$

On R-points the inclusions

$$\widehat{\mathrm{symm}}_n \to \widehat{\mathrm{symm}}^0$$

take $f(x)$ to $x^{-n} f(x)$. Since filtered colimits commute with finite products the compatible system of maps

$$\begin{array}{ccc} \widehat{\mathrm{symm}}_i \times \widehat{\mathrm{symm}}_j & \xrightarrow{\oplus_{i,j}} & \widehat{\mathrm{symm}}_{i+j} \\ \downarrow{\scriptstyle \iota \times \iota} & & \downarrow{\scriptstyle \iota \circ \iota} \\ \widehat{\mathrm{symm}}_{i+1} \times \widehat{\mathrm{symm}}_{j+1} & \xrightarrow{\oplus_{i+1,j+1}} & \widehat{\mathrm{symm}}_{i+j+2} \end{array}$$

defines an operation

$$\oplus : \widehat{\mathrm{symm}}^0 \times \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}^0$$

which we combine with

$$i_0 : \widehat{\mathrm{symm}}_0 \to \widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}^0$$

to make $\widehat{\mathrm{symm}}^0$ into a formal monoid scheme. The product on $\widehat{\mathrm{symm}}^0$ corresponds to multiplication of polynomials.

Moreover, polynomials whose constant coefficient is one and other coefficients are nilpotent form a group under multiplication. Indeed, the multiplicative inverse of

$$f(z) = 1 + \sum_{i=1}^{n} b_i z^i$$

is a power series where the coefficient of z^{n+k} lies in

$$(b_1, \ldots, b_n)^k.$$

Since $b_i \in \mathrm{Nil}(R)$ this ideal is zero for large k and we see that $1/f$ is actually a polynomial of the correct form. It follows that $\widehat{\mathrm{symm}}^0$ is a formal group scheme.

Definition 3.15. *The positive symmetric scheme is the scheme*

$$\widehat{\mathrm{symm}}^+ = \coprod_{i \geq 0} \widehat{\mathrm{symm}}_i = \mathrm{colim} \coprod_{0 \leq i \leq n} \widehat{\mathrm{symm}}_i$$

equipped with the rig structure defined by the maps $\oplus_{i,j}$ and $\otimes_{i,j}$ with the additive identity given by the inclusion $\widehat{\mathrm{symm}}_0 \to \widehat{\mathrm{symm}}^+$ and the multiplicative identify given by i_0 followed by the inclusion $\widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}^+$.

Definition 3.16. *Assembling the maps*

$$\iota : \widehat{\mathrm{symm}}_i \to \widehat{\mathrm{symm}}_{i+1}$$

into a map $\mathrm{symm}^+ \to \widehat{\mathrm{symm}}^+$, *we set*

$$\widehat{\mathrm{symm}} \equiv \mathrm{colim} \left[\widehat{\mathrm{symm}}_+ \to \widehat{\mathrm{symm}}_+ \to \ldots \right].$$

Note that if we restrict to the 0 component of $\widehat{\mathrm{symm}}_+$ and then take colimits we obtain the same system defining $\widehat{\mathrm{symm}}^0$. Hence the zeroth component of $\widehat{\mathrm{symm}}$ is $\widehat{\mathrm{symm}}^0$ and this component inherits a multiplication from $\widehat{\mathrm{symm}}$.

Remark 3.17. There is nothing special about this multiplication. We can take any group operation on $\widehat{\mathrm{symm}}_1$ with unit i_0 and extend it to define rig schemes, ring schemes and rng schemes.

A formal group structure

$$F : \widehat{\mathrm{symm}}_1 \times \widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}_1$$

determines a map

$$F^s : \widehat{\mathrm{symm}}^+ \times \widehat{\mathrm{symm}}^+ \to \widehat{\mathrm{symm}}^+$$

that makes $\widehat{\mathrm{symm}}^+$ into a formal rig scheme. Using Construction 3.11 we can define maps

$$F_{i,j} : \widehat{\mathrm{symm}}_i \times \widehat{\mathrm{symm}}_j \to \widehat{\mathrm{symm}}_{ij}.$$

By construction these maps fit together and distribute over addition. This multiplicative inherits its unital and associativity properties from F.

The colimit in 3.16 can be identified with Grothendieck's group completion construction which makes $\widehat{\mathrm{symm}}$ into a formal ring scheme. After restricting this multiplicative structure to the 0 component we obtain:

Proposition 3.18. *A formal group structure*

$$F : \widehat{\mathrm{symm}}_1 \times \widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}_1$$

determines a map

$$F^s : \widehat{\mathrm{symm}}^0 \times \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}^0$$

that makes $\widehat{\mathrm{symm}}^0$ *into a formal rng scheme.*

Since the endomorphisms of a commutative group object always form a (generally non-commutative) ring, every group object has a $\mathbb{Z}$-module structure. Under this $\mathbb{Z}$-action on a group object G with multiplication μ, a positive integer n corresponds to the composite

$$[n] : G \xrightarrow{\Delta^{n-1}} G^n \xrightarrow{\mu^{n-1}} G.$$

Proposition 3.19. *Let n be a positive integer and G a connected (formal) commutative group scheme over a ring R containing $\mathbb{Z}[1/n]$. Then the $\mathbb{Z}$-module structure described above extends to a $\mathbb{Z}[1/n]$-module structure.*

Proof. It suffices to show that $[n]$ is an isomorphism. On the Hopf-algebra representing G we see that, modulo decomposables, $[n]$ takes any indecomposable to n times itself (connectivity of our Hopf algebra is key here). Since n is invertible over our base ring, this map is an isomorphism. □

If $n = kl$ then $[n]$ factors as $[k] \circ [l]$. In the case $G = \widehat{sym}^0$, we can still obtain a nontrivial factorization when n is a prime p: $[p] = V_p \circ F_p$. Where V_p and F_p are the following:

Definition 3.20. *Let the kth Frobenius operation $F_k : \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}_0$ be the unique endomorphism satisfying*

$$F_k : (1 - az) \mapsto (1 - az^k).$$

Remark 3.21. We can formally factorize any degree n polynomial in $\widehat{\mathrm{symm}}^0$ into a product of linear factors like the above. So this map is indeed determined by its behavior on a linear term. Equivalently, we could have constructed this map using Theorem 3.9.

Definition 3.22. *Let the kth Verschiebung operation $V_k : \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}_0$ be the unique endomorphism satisfying*

$$V_k : (1 - az) \mapsto (1 - a^k z).$$

3.2. **Lambda schemes.** In this section we will examine the scheme Λ and its dual. The scheme Λ plays an important role in the theory of λ-rings which encode common structures in representation theory and algebraic topology, see [AT69, Knu73]).

Definition 3.23. *The Lambda-scheme Λ is the ring scheme whose underlying additive group scheme is defined by*

$$\Lambda(R) = (1 + tR[\![z]\!])^\times.$$

The multiplicative structure is more complicated and we will explain it below. This scheme can be represented by the ring $\mathrm{Sym} = \mathbb{Z}[b_1, b_2, \dots]$, *since a homomorphism* $f : \mathrm{Sym} \to R$ *is determined by where the b_i are mapped to under f. These elements determine a power series*

$$\sum_{i \geq 0} f(b_i) z^i, \tag{3.24}$$

where we adopt the useful convention $b_0 = 1$ *and therefore* $f(b_0) = 1$*. Under this correspondence the additive group is described by*

$$\begin{array}{rcllcl} \epsilon_+ : \mathrm{Sym} & \to & \mathbb{Z} & b_n & \mapsto & 0 \\ \Delta_+ : \mathrm{Sym} & \to & \mathrm{Sym} \otimes \mathrm{Sym} & b_n & \mapsto & \sum_{i=0}^{n} b_i \otimes b_{n-i} \\ \chi_+ : \mathrm{Sym} & \to & \mathrm{Sym} & b_n & \mapsto & -\sum_{i=0}^{n-1} \chi_+(b_i) b_{n-i} \end{array}$$

for all $n \geq 1$*.*

Since Λ takes values in formal rings we might expect it to be an inverse limit of regular schemes and in fact it is. Since $\mathrm{Sym} \cong \operatorname{colim} \mathrm{Sym}_n$ where $\mathrm{Sym}_n = \mathbb{Z}[b_1, \ldots, b_n]$, we obtain

$$\mathrm{symm} = \operatorname{Spec} \mathrm{Sym} = \mathcal{R}ing(\operatorname{colim} \mathrm{Sym}_n, -) \cong \lim \mathcal{R}ing(\mathrm{Sym}_n, -) = \lim \operatorname{Spec} \mathrm{Sym}_n.$$

While it is clear that the Λ is an informal analogue of $\widehat{\mathrm{symm}}^0$, we have the following result whose proof we defer until Section 4.

Theorem 3.25. *The Cartier dual of* $\widehat{\mathrm{symm}}^0$ *is* Λ*.*

The Frobenius and Verschiebung maps defined above on $\widehat{\mathrm{symm}}^0$ induce maps on the Cartier dual. The conventional names for the duals of F_k and V_k are V_k and F_k respectively, i.e., the Frobenius and Verschiebung are interchanged under Cartier duality. The reader can check that this definition agrees with the obvious informal analogues of Definitions 3.20 and 3.22.

3.3. **Witt schemes.** Witt schemes appear in many areas of mathematics, from starring roles in the classification of commutative group schemes and p-divisible groups ([Dem72]), to class field theory for fields of characteristic p (Witt's original purpose) and to cameo appearances in the Teichmuller embedding of finite fields into rings of characteristic zero. The role that Witt schemes (or more precisely, the truncated Witt schemes), play in commutative group schemes is reflected in the classification of bicommutative Hopf algebras and their characterization by Dieudonne modules ([Goe98, Sch70]) . Their role in constructing characteristic 0 lifts of finite fields leads to their appearance in the construction of Landweber exact formal group laws and their associated cohomology theories ([Rez98]).

The Witt scheme is a ring scheme whose underlying scheme is isomorphic to $\mathbb{A}^\infty$, just like Λ. In fact, there is an isomorphism of ring schemes between them. We will exploit this fact to circumvent defining the Witt scheme's ring structure directly and save us the trouble of restating a number of integrality lemmas (see [Haz78]). On representing rings, this isomorphism reflects a different choice of generators which are more convenient for some purposes. For example, the formulas for the primitive elements are simpler and satisfy some useful congruences. In Section 5 we will give formulas relating the choices of generators.

Definition 3.26. *The Witt scheme* $\mathbb{W}$ *has the underlying scheme* $\operatorname{Spec}(\mathbb{Z}[\theta_1, \theta_2 \ldots])$*, and a ring scheme structure which will be given in Corollary 3.28. We identify an element* $f \in \mathbb{W}(R)$ *with the power series*

$$\prod (1 - f(\theta_n) t^n)^{-1} = 1 + p_1(f) t + p_2(f) t^2 + \cdots \in 1 + tR[\![t]\!].$$

For example,

$$\begin{array}{rcl} p_1(f) & = & f(\theta_1) \\ p_2(f) & = & f(\theta_1)^2 + f(\theta_2) \\ p_3(f) & = & f(\theta_1)^3 + f(\theta_1) f(\theta_2) + f(\theta_3). \end{array}$$

Examining these formulas for the coefficients and comparing them to Equation 3.24 defining the R-points of Λ, we can find a formula for $f(\theta_i)$ in terms of the $f(b_i)$, and conversely, inductively. This leads us to the following theorem :

Theorem 3.27 (cf. [Haz78])**.** *There is an isomorphism of schemes* $\mathbb{W} \cong \Lambda$.

Proof. By the Yoneda lemma, the maps from $\mathbb{W}$ to Λ are in bijection with

$$\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots]) \cong \mathcal{R}ing(\mathbb{Z}[b_1, b_2, \dots], \mathbb{Z}[\theta_1, \theta_2, \dots]).$$

The power series

$$\prod (1 - \theta_n t^n)^{-1}$$

defines an element of $\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots])$ and hence a map f. This map gives rise to maps

$$f_n : \mathbb{Z}[b_1, \dots, b_n] \to \mathbb{Z}[\theta_1, \dots, \theta_n].$$

Each of these algebras admits an augmentation that sends each of the polynomial generators to 0. The induced map on indecomposables

$$f_n : (b_1, \dots, b_n)/(b_1, \dots, b_n)^2 \to (\theta_1, \dots, \theta_n)/(\theta_1, \dots, \theta_n)^2$$

is an isomorphism because we have the following isomorphism modulo decomposables

$$\begin{aligned} \prod_{1 \le i \le n} (1 - \theta_i t^i)^{-1} &\equiv \prod_{1 \le i \le n} (1 + \theta_i t^i) \\ &\equiv 1 + \sum_{1 \le i \le n} \theta_i t^i. \end{aligned}$$

Using powers of the augmentation ideals to define filtrations on these algebras, we have an induced isomorphism on the associated graded algebras

$$gr(\mathbb{Z}[b_1, \dots, b_n]) \to gr(\mathbb{Z}[\theta_1, \dots, \theta_n]).$$

With such a filtration the associated graded of a polynomial algebra is itself, so we have an isomorphism between the truncated algebras $\mathbb{Z}[b_1, \dots, b_n]$ and $\mathbb{Z}[\theta_1, \dots, \theta_n]$. We obtain the desired isomorphism by taking colimits. □

Corollary 3.28. *The Witt scheme $\mathbb{W}$ admits the structure of a ring-scheme such that the map in Theorem 3.27 is an isomorphism of ring-schemes.*

Remark 3.29. We can consider this to be a definition of the ring-scheme structure on $\mathbb{W}$.

Corollary 3.30. *There is an isomorphism of formal group schemes* $\widehat{\mathbb{W}} \cong \widehat{\mathrm{symm}}^0$.

4. Formal schemes arising from the cohomology of a space

Now we will try to apply the above theory to the co/homology of a space. We are particularly interested in those spaces and cohomology theories that are connected to formal groups (see [Ada95, Hop99, Rav00]).

Notation 4.1. *If E is a cohomology theory and X a space then $E^*(X)$ will always refer to the unreduced E-cohomology of X. The reduced cohomology theory will be denoted $\tilde{E}^*(X)$.*

Notation 4.2. *For a multiplicative cohomology theory E, we will adopt the standard convention of writing E_* for $E^{-*}(*)$.*

Definition 4.3. *A cohomology theory E is called even-periodic if*

(1) *E is multiplicative (i.e., E takes values in graded rings).*
(2) $E_{\mathrm{odd}} = 0$
(3) *There exists a unit* $x \in E_2$.

The standard examples include even-periodic ordinary cohomology HPR, complex K-theory K, even-periodic Morava K-theory $\overline{K(n)}$, the Morava E-theories E_n, and even-periodic complex cobordism MP. We typically recognize these theories by their coefficient rings:

$$\begin{array}{rcl} HPR_* & \cong & R[v, v^{-1}] \\ K_* & \cong & \mathbb{Z}[v, v^{-1}] \\ \overline{K(n)}_* & \cong & \mathbb{F}_{p^n}[v, v^{-1}] \\ E_{n*} & \cong & \mathbb{W}_p(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][v, v^{-1}] \text{ (see Section 3.3)} \\ MP_* & \cong & \mathbb{Z}[b_1, b_2, \ldots][v, v^{-1}], \end{array}$$

where the grading is determined by putting all of the generators in degree 0 except for v which lies in degree 2. A nice description of the properties of these cohomology theories can be found in [Hop99, Rez98].

For the remainder of this chapter E will always denote some even-periodic cohomology theory.

Recall that the tensor product operation on vector bundles restricts to give a group operation on isomorphism classes of line bundles, the unit coming from the one dimensional trivial bundle [1] and the inverse of a bundle η is given by the dual bundle η^*. Since $\mathbb{C}P^\infty$ is a model for $BU(1)$ the classifying space of 1-dimensional complex line-bundles we obtain a multiplication map

$$\mu_\otimes : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

that makes $\mathbb{C}P^\infty$ into a group object in *hTop*, the derived category of topological spaces.

Standard calculations (see [Hop99]) show that for an even-periodic cohomology theory E, we have

$$E^0(\mathbb{C}P^\infty) \cong E_0[\![x]\!] \tag{4.4}$$

with the choice of isomorphism dependent on the choice of unit in Definition 4.3. We also have Kunneth isomorphisms

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E^0(\mathbb{C}P^\infty) \otimes_{E_0} \cdots \otimes_{E_0} E^0(\mathbb{C}P^\infty).$$

Fixing an isomorphism as in Equation 4.4 canonically determines an isomorphism

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E_0[\![t_1, \ldots, t_n]\!].$$

The map $\mu_\otimes$ gives rise to the coproduct

$$\begin{array}{rcl} \Delta_\otimes : E^0(\mathbb{C}P^\infty) & \longrightarrow & E^0(\mathbb{C}P^\infty) \otimes_{E_0} E^0(\mathbb{C}P^\infty) \\ E^0[\![x]\!] & \longrightarrow & E^0[\![x, y]\!] \\ x & \mapsto & F_E(x, y). \end{array}$$

The formal power series $F_E(x, y)$ is the formal group law associated to E with a specified orientation (which determines the isomorphism in Equation 4.4). A different choice of isomorphism will give rise to a formal group law of the form $F_E(\lambda x, \lambda y)$, for some unit $\lambda \in E^0$.

Although the power series

$$\Delta_\otimes(x) = F_E(x, y)$$

is called a formal group law, the map $\Delta_\otimes$ actually defines a cogroup object in pro-rings. Passing to the opposite category of formal schemes allows us to reverse the arrows and recover a group object.

Definition 4.5. *Given a CW-complex X and an even-periodic cohomology theory E, we define the formal scheme X_E associated to X and E by*

$$X_E = \operatorname{colim} \operatorname{Spec} E^0(X_\alpha),$$

where the filtered system defining the colimit is given by the filtration of X by its finite subcomplexes X_α.

Remark 4.6. Note that X_E is a covariant functor of X, making the notation convenient for studying diagrams of spaces. Also note that X_E must not be confused with the Bousfield localization $X_{\langle E \rangle}$, of X with respect to a homology theory E.

If X is a finite-dimensional CW-complex then X_E is defined by a directed system with terminal object $\operatorname{Spec} E^0(X)$. It follows that X_E is isomorphic to the ordinary scheme $\operatorname{Spec} E^0(X)$.

Definition 4.7. *The formal group associated to an even-periodic cohomology theory E, $\widehat{\mathbb{G}}_E$ is the formal scheme $\mathbb{C}P^\infty_E$ over E_0, with the group structure induced by the tensor product of vector bundles.*

By well known calculations we can identify the formal group associated to K-theory with $\widehat{\mathbb{G}}_m$ from Example 2.9 and the formal group associated to ordinary cohomology $HP\mathbb{Z}$ with $\widehat{\mathbb{G}}_a$ from Example 2.8.

If X is a commutative H-group with $H_*(X)$ even and torsion-free then the relevant Atiyah-Hirzebruch and Kunneth spectral sequences collapse to show

$$\begin{aligned} E_*(X) &\cong E_* \otimes H_*(X) \\ E_*(X \times X) &\cong E_*(X) \otimes_{E_*} E_*(X) \end{aligned}$$

From this we can see that the H-group structure on X makes $\operatorname{Spec} E_* X$ a group scheme.

Since $H_* X$ is torsion-free and of finite type then similar arguments show

$$E^*(X) \cong \operatorname{Mod}_{E_*}(E_* X, E_*).$$

It now follows that:

Proposition 4.8. *Suppose X is a commutative H-group with $H_*(X)$ even, torsion free, and of finite type. Then $D \operatorname{Spec} E_0 X \cong X_E$.*

Now we recall several well known calculations (see, for example, [Swi02]).

Proposition 4.9. *The inclusion of a maximal torus*

$$(S^1)^{\times n} \to U(n)$$

induces a map

$$E^*(BU(n)) \to E^*((BS^1)^{\times n}) \cong E^*((\mathbb{C}P^\infty)^{\times n}) \cong E^*[\![t_1, \ldots, t_n]\!]$$

which lifts to an isomorphism

$$E^*(BU(n)) \cong E^*((\mathbb{C}P^\infty)^{\times n})^{\Sigma_n}.$$

Example 4.10. Combining this with Theorem 3.6 we see that $E^* BU(n) \cong E^*[\![\sigma_1, \ldots, \sigma_n]\!]$. It follows that we can identify $BU(n)_E$ with $\widehat{\operatorname{symm}}_n \times \operatorname{Spec} E^0$.

Proposition 4.11 ([AGP02, May99])**.** *The functor $K^0(-)$ is represented by the space $BU \times \mathbb{Z}$.*

The maps from $BU(i)$ into $BU(i+1)$, for each i, that classify adjoining a one dimensional trivial bundle to the universal i-dimensional bundle, induce a map

$$\coprod_{i\geq 0} BU(i) \to \coprod_{i\geq 0} BU(i)$$

that can be used to construct the following homotopy equivalence:

$$BU \times \mathbb{Z} \simeq \operatorname{hocolim} \left[\coprod_{i\geq 0} BU(i) \to \coprod_{i\geq 0} BU(i) \to \coprod_{i\geq 0} BU(i) \to \dots \right].$$

Example 4.12. Combining Proposition 4.11 with 4.10 we see that we can identify $(BU \times \mathbb{Z})_E$ with $\widehat{\mathrm{symm}} \times \operatorname{Spec} E^0$ and $BU_E \equiv (BU \times \{0\})_E$ with $\widehat{\mathrm{symm}}^0 \times \operatorname{Spec} E^0$.

Using the construction of $\widehat{\mathrm{symm}}^0$ as the colimit of the $\widehat{\mathrm{symm}}_n$, or from the construction of BU given above, we see that

$$E^* BU \cong E^*[\![\sigma_1, \sigma_2, \dots]\!].$$

Since BU satisfies the conditions of Proposition 4.8, we see that the Cartier dual of $\widehat{\mathrm{symm}}^0$ is represented by $\operatorname{Spec}(E_0 BU)$. It is well known that, up to completion, the homology of BU is self-dual as a Hopf algebra. It follows that $\operatorname{Spec}(E_0 BU)$ is the informal analogue of $\widehat{\mathrm{symm}}^0$ and

$$\operatorname{Spec}(E_0 BU) \cong \Lambda \times \operatorname{Spec} E^0.$$

We can now consolidate the work above.

Theorem 4.1. *If E is an even-periodic ring spectrum then we have the following chain of group scheme isomorphisms:*

$$\begin{aligned} \operatorname{Spec} E_0 BU &\cong \Lambda \times \operatorname{Spec} E^0 \\ &\cong \mathbb{W} \times \operatorname{Spec} E^0. \end{aligned}$$

We also have the following chain of formal group scheme isomorphisms:

$$\begin{aligned} BU_E &\cong \widehat{\mathrm{symm}}^0 \times \operatorname{Spec} E^0 \\ &\cong \widehat{\mathbb{W}} \times \operatorname{Spec} E^0 \equiv D\mathbb{W} \times \operatorname{Spec} E^0 \\ &\cong \widehat{\Lambda} \times \operatorname{Spec} E^0 \equiv D\Lambda \times \operatorname{Spec} E^0. \end{aligned}$$

Remark 4.13. We could have extended this chain of isomorphisms to connnect to the Burnside ring of $\widehat{\mathbb{Z}}$, the necklace algebra, or to the curves functor [Haz08] if it were not for constraints on time, space, and energy. These correspondences are too beautiful and common to avoid being rediscovered again and again; this is especially true for the author. The extensive bibliography (471 entries!) in [Haz08] is a testament to this.

We conclude this section with a simple application of the above theorem. When working of p-local ring the Cartier dual of the Witt scheme admits the following idempotent map of group schemes:

$$\epsilon = \sum_{\gcd(n,p)=1}^{\widehat{\mathbb{W}}} \left[\frac{\mu(n)}{n} \right] V_n F_n \tag{4.14}$$

where $\mu(n)$ is the Möbius function defined by the recurrence relation

$$\sum_{d|n} \mu(d) = \delta_{1,n}.$$

This idempotent map splits $\widehat{\mathbb{W}}$ into a countable product of group schemes. The image of the idempotent is denoted $\widehat{\mathbb{W}}_p$ which is Cartier dual to the p-Witt scheme.

This determines a splitting of each of the formal group schemes in Theorem 4.1. In particular, it determines a Hopf algebra splitting of E^*BU that corresponds to the generalized cohomology form of Husemoller's splitting [Hus71]. Applying Cartier duality we obtain a splitting of E_*BU and the image of the idempotent is self-dual (up to completion).

The essential part of the construction of this splitting, is the identification of E_*BU with the free commutative algebra $\tilde{E}_*(\mathbb{C}P^\infty)$. This is dual to the statement that E^*BU is the cofree coalgebra on $\tilde{E}^*(\mathbb{C}P^\infty)$, which is a reformulation of the splitting principle for E^*BU.

One can also see that Equation 4.14 is exactly the formula for Quillen's idempotent [Qui69, 7] on curves in a formal group, which he used to split $MU_{(p)}$. Indeed both splittings are constructed in the same way.

5. The Chern classes of a tensor product of arbitrary vector bundles

Suppose we have two 3-dimensional complex vector bundles over some fixed space which have E-theory Chern classes a_1, a_2, a_3 and b_1, b_2, b_3 respectively. The tensor product of these two bundles is a 9-dimensional vector bundle and this operation defines a map of algebras:

$$\Delta : E^*(BU(9)) \cong E^*[\![c_1, \ldots, c_9]\!] \to E^*[\![a_1, a_2, a_3]\!] \otimes_{E^*} E^*[\![b_1, b_2, b_3]\!].$$

By our form of the splitting principle (Theorem 3.9), we obtain formulas for this map (Δ corresponds to $\otimes_{3,3}$). Even for such a small example the formulas already are quite complicated. Computing the coproduct of higher Chern classes is greatly facilitated by using a computer and we have implemented our calculations in Mathematica, although it is straightforward to implement them in any symbolic computer package. In the first two examples below, we have grouped the terms together to emphasize the symmetry in the expansions.

Due to obvious limitations on space and the reader's assumed interest, we have only included the first few coproducts in each of the cases below. The author is not aware of such formulas ever appearing in print and we record them for posterity, although our primary interest is in demonstrating that through a limited range these formulas are computable by the above methods.

The simplest possible example is when E is $H\mathbb{Z}$. In this case we are working with the additive formal group law described above. In this particular case, one can find these formulas (in an unexpanded form) in [MS74, p. 87-88].

$$
\begin{aligned}
\Delta c_1 =&\ 3\,(a_1 \otimes 1 + 1 \otimes b_1) \\
\Delta c_2 =&\ 3\left((a_2 + a_1^2) \otimes 1 + 1 \otimes (b_2 + b_1^2)\right) + 8 a_1 \otimes b_1 \\
\Delta c_3 =&\ (a_1^3 + 6a_1a_2 + 3a_3) \otimes 1 + 1 \otimes (b_1^3 + 6b_1b_2 + 3b_3) \\
&+ 7\left(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\right) \\
\Delta c_4 =&\ + 3\left((a_1^2a_2 + a_2^2 + 2a_1a_3) \otimes 1 + 1 \otimes (b_2^2 + b_1^2b_2 + 2b_1b_3)\right) \\
&+ 2\left((6a_1a_2 + 3a_3 + a_1^3) \otimes b_1 + a_1 \otimes (6b_1b_2 + 3b_3 + b_1^3)\right) \\
&+ 3a_2 \otimes b_2 + 5a_1^2 \otimes b_1^2 \\
&+ 6\left(a_1^2 \otimes b_2) + a_2 \otimes b_1^2\right)
\end{aligned}
$$

When $E = KU$ is equipped with the (E_∞) orientation described above we obtain the following coproduct formulas (here $u \in K^{-2}$ is the Bott element):

$$
\begin{aligned}
\Delta c_1 =&3(a_1 \otimes 1 + 1 \otimes b_1) - u a_1 \otimes b_1 \\
\Delta c_2 =&3\left((a_1^2 + a_2) \otimes 1 + 1 \otimes (b_1^2 + b_2)\right) + 8a_1 \otimes b_1 \\
&- 2u\left(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\right) \\
&+ u^2\left(a_1^2 \otimes b_2 - 2a_2 \otimes b_2 + a_2 \otimes b_1^2\right) \\
\Delta c_3 =&a_1^3 \otimes 1 + 1 \otimes b_1^3 + 6(a_1a_2 \otimes 1 + 1 \otimes b_1b_2) + 3(a_3 \otimes 1 + 1 \otimes b_3) \\
&+ 7\left(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\right) \\
&+ u\left[-(a_1^3 + 6a_1a_2 + 3a_3) \otimes b_1 - a_1 \otimes (b_1^3 + 6b_1b_2 + 3b_3)\right. \\
&\quad \left. -2(a_1^2 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1^2) - 8a_2 \otimes b_2\right] \\
&+ u^2\left[a_1^3 \otimes b_2 + a_2 \otimes b_1^3 + 2(a_1a_2 \otimes b_1^2 + a_1^2 \otimes b_1b_2)\right. \\
&\quad \left. +3(a_3 \otimes b_1^2 - 2a_2 \otimes b_3 - 2a_3 \otimes b_2 + a_1^2 \otimes b_3)\right] \\
&+ u^3\left[-(a_1^3 \otimes b_3 + a_1a_2 \otimes b_1b_2 + a_3 \otimes b_1^3)\right. \\
&\quad \left. +3(a_1a_2 \otimes b_3 - a_3 \otimes b_3 + a_3 \otimes b_1b_2)\right]
\end{aligned}
$$

The above two complex oriented theories have the only formal group laws (up to isomorphism) that are finite. When working with another theory we are forced to use finite expansions.

The formulas below are based on an expansion of the formal group law for BP, at the prime 3, out to the 4th power in the first Chern class. We have chosen to use the Hazewinkel generators since they provide a formal group law with *integral* coefficients. Using these choices we obtain the following coproduct formulas for the tensor product of two 2-dimensional vector bundles.:

$$\begin{aligned}
\Delta c_1 =& 2\left(a_1 \otimes 1 + v_1(a_1 \otimes b_2 + a_2 \otimes b_1) + 1 \otimes b_1\right) \\
& - v_1\left(a_1 \otimes b_1^2 + a_1^2 \otimes b_1\right) \\
\Delta c_2 =& - 5v_1^2 a_1 a_2 \otimes b_1 b_2 - v_1 a_1 \otimes b_1^3 + v_1 a_1 a_2 \otimes b_1 + v_1 a_1 \otimes b_1 b_2 \\
& + v_1^2 a_1 a_2 \otimes b_1^3 - 2v_1 a_1^2 \otimes b_1^2 + v_1^2 a_1^2 \otimes b_2^2 - v_1 a_1^3 \otimes b_1 \\
& + v_1^2 a_1^3 \otimes b_1 b_2 + v_1^2 a_1^4 \otimes b_2 - 4v_1^2 a_2 a_1^2 \otimes \ b_2 + 8v_1 a_2 \otimes b_2 \\
& - 4v_1^2 a_2 \otimes b_1^2 b_2 + 2v_1^2 a_2 \otimes b_2^2 + v_1^2 a_2 \otimes b_1^4 + 2v_1^2 a_2^2 \otimes b_2 \\
& + v_1^2 a_2^2 \otimes b_1^2 + 3a_1 \otimes b_1 + a_1^2 \otimes 1 + 2a_2 \otimes 1 + 1 \otimes b_1^2 + 2 \otimes b_2 \\
\Delta c_3 =& - v_1 a_1 a_2 \otimes b_1^2 + 2v_1 a_1 a_2 b_2 + 2v_1^2 a_1 a_2 \otimes b_2^2 \\
& - v_1^3 a_1 a_2 \otimes b_1^2 b_2^2 + 2v_1^3 a_1 a_2 \otimes b_2^3 - 4v_1^2 a_1 a_2 \otimes b_1^2 b_2 \\
& - 6v_1^2 a_1 a_2^2 \otimes b_2 + 2v_1^3 a_1 a_2^2 \otimes b_1^2 b_2 + a_1 \otimes b_1^2 + 2a_1 \otimes b_2 \\
& - 2v_1 a_1 \otimes b_1^2 b_2 + 2v_1 a_1 \otimes b_2^2 + v_1^2 a_1 a_2 \otimes b_1^4 - v_1^3 a_1^2 a_2^2 \otimes b_1 b_2 \\
& + a_1^2 \otimes b_1 - v_1 a_1^2 \otimes b_1^3 - v_1 a_1^2 b_1 b_2 + v_1^2 a_1^2 a_2 \otimes b_1^3 + v_1^2 a_1^2 \otimes b_1 b_2^2 \\
& + 2v_1^3 a_1^3 a_2 \otimes b_2^2 - v_1 a_1^3 \otimes b_1^2 + v_1^2 a_1^3 \otimes b_1^2 b_2 + v_1^2 a_1^4 \otimes b_1 b_2 \\
& - 2v_1 a_2 \otimes a_1^2 \otimes b_1 - 4v_1^2 a_2 a_1^2 \otimes b_1 b_2 - v_1^3 a_1^2 a_2 \otimes b_1^3 b_2 \\
& + 2v_1^3 a_1^2 a_2 \otimes b_1 b_2^2 + 2v_1^2 a_1^3 a_2 \otimes b_2 - v_1^3 a_1^3 a_2 \otimes b_1^2 b_2 \\
& + 2a_2 \otimes b_1 + 2v_1 a_2 \otimes b_1 b_2 + 2v_1^2 a_2 \otimes b_1^3 b_2 \\
& - 6v_1^2 a_2 \otimes b_1 b_2^2 - 6v_1^3 a_1 a_2^2 \otimes b_2^2 + 2v_1 a_2^2 \otimes b_1 + 2v_1^2 a_2^2 \otimes b_1 b_2 \\
& + 2v_1^3 a_2^2 \otimes b_1^3 b_2 - 6v_1^3 a_2^2 \otimes b_1 b_2^2 + v_1^2 a_1 a_2^2 \otimes b_1^2 \\
& + 2v_1^3 a_2^3 \otimes b_1 b_2 + 2a_1 a_2 \otimes 1 + 2 \otimes b_1 b_2
\end{aligned}$$

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